MATHEMATICS GRADE 12
MHF4U, ADVANCED FUNCTIONS

Date SUMMER 2008

TEST 2 FUNCTIONS

Teacher Teodoru Gugoiu

Name ....

Total $\left[\frac{\quad}{50}\right]$

1. Consider the function $f(x)=\sqrt{\dfrac{x-1}{x+1}}$. Find: [K/U 2 marks]

a) $f(2)=$
$=\sqrt{\dfrac{2-1}{2+1}}$
$=\sqrt{\dfrac{1}{3}}=\dfrac{1}{\sqrt{3}}$

b) $f(0)=$
$=\sqrt{\dfrac{0-1}{0+1}}=\sqrt{\dfrac{-1}{1}}=$
$=\sqrt{-1}$
not defined

c) $f\left(\dfrac{1}{a}\right)=$
$=\sqrt{\dfrac{\frac{1}{a}-1}{\frac{1}{a}+1}}=$
$=\sqrt{\dfrac{1-a}{1+a}}$

d) $f(a+1)=$
$=\sqrt{\dfrac{(a+1)-1}{(a+1)+1}}=$
$=\sqrt{\dfrac{a}{a+2}}$

2. Find the domain and the range of each function defined by a set of ordered pairs. [K/U 2 marks]

a) $f=\{(3,1),(-2,2),(-3,2),(2,1)\}$

$D=\{3,-2,-3,2\}$
$R=\{1,2\}$

b) $f=\{(1,0),(0,1),(-1,2),(5,0),(-2,-2)\}$

$D=\{1,0,-1,5,-2\}$
$R=\{0,1,2,-2\}$

3. Find the domain and the range of each function. [K/U 3 marks]

a) $y=5-(2x+3)^2$

$D=(-\infty,\infty)$
$R=\{y\in R \mid y\le 5\}$
$=(-\infty,5]$

b) $y=3-\sqrt{x^2-x-6}$
$=3-\sqrt{(x-3)(x+2)}$

x		-2		3	
x-3	- -	-	-	0	+ + + +
x+2	- -	0	+ +	+	+ + + +
all	+ +	0	- -	0	+ + + +

$D=(-\infty,-2]\cup[3,\infty)$
$R=(-\infty,3]$

c) $y=\dfrac{1}{(x-2)^2}-2$

$D=\{x\in R \mid x\neq 2\}$
$=(-\infty,2)\cup(2,\infty)$
$R=(-2,\infty)$

4. Consider the following functions $f(x)=\sqrt{x-1}$, $g(x)=x^2+2$, and $h(x)=\dfrac{1}{x-3}$. Find: [K/U 4 marks]

a) $(f\circ g)(2)$
$=f(g(2))=f(6)=\sqrt{5}$

b) $(h\circ f)(5)=h(f(5))=h(2)=$
$=\dfrac{1}{2-3}=-1$

c) $(f\circ g\circ h)(4)-(h\circ g\circ f)(4)$
$=f(g(h(4)))-h(g(f(4)))=$
$=f(g(1))-h(g(\sqrt{3}))=f(3)-h(5)=\sqrt{2}-\dfrac{1}{2}$

5. Find the inverse function of the following functions: [K/U 3 marks]

a) $f(x)=2x-3$
$y=2x-3$
$x=2y-3$
$y=\dfrac{x+3}{2}$
$f^{-1}(x)=\dfrac{x+3}{2}$

b) $f(x)=2-\sqrt{x+1}$
$y=2-\sqrt{x+1}$
$x=2-\sqrt{y+1}$
$\sqrt{y+1}=2-x$
$y+1=(2-x)^2$
$y=(2-x)^2-1$
$f^{-1}(x)=(2-x)^2-1$

c) $f(x)=\dfrac{x-2}{x+2}$
$y=\dfrac{x-2}{x+2}$
$x=\dfrac{y-2}{y+2}$
$xy+2x=y-2$
$(x-1)y=-2x-2$
$f^{-1}(x)=\dfrac{-2x-2}{x-1}$

6. Find two functions $f(x)$ and $g(x)$ such that $(f \circ g)(x) = \dfrac{x^2}{x^2+1} - 1$. [K/U 2 marks]

$g(x) = x^2$

$f(x) = \dfrac{x}{x+1} - 1$

check: $(f \circ g)(x) =$

$= f(g(x)) = f(x^2) = \dfrac{x^2}{x^2+1} - 1$

7. Classify each function as even, odd or neither. Show your work. [K/U 3 marks]

a) $f(x) = x + \dfrac{1}{x-1}$

$f(-x) = -x + \dfrac{1}{-x-1} =$

$= -\left(x + \dfrac{1}{x+1}\right)$

$f(-x) \neq f(x)$

$f(-x) \neq -f(x)$

$\therefore$ neither

b) $g(x) = x^4 - \dfrac{1}{x^2 + |x|}$

$g(-x) = (-x)^4 - \dfrac{1}{(-x)^2 + |-x|} =$

$= x^4 - \dfrac{1}{x^2+|x|} = g(x)$

$g(-x) = g(x)$

$\therefore$ $g(x)$ is even

c) $h(x) = \dfrac{-x^2}{2x^3 - 2x}$

$h(-x) = \dfrac{-(-x)^2}{2(-x)^3 - 2(-x)} =$

$= \dfrac{-x^2}{-2x^3+2x} = -\dfrac{-x^2}{2x^3-2x} =$

$= -h(x)$

$\therefore$ $h(x)$ is odd

8. Consider the functions $f(x) = \dfrac{1}{x^2} - 2$ and $g(x) = \sqrt{x-4}$. Find $(f \circ g)(x)$ and state its domain and range. [A 4 marks]

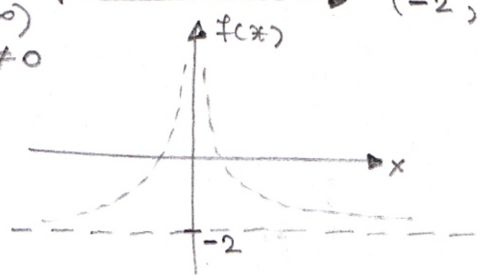
$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-4}) = \dfrac{1}{x-4} - 2$

$[4, \infty) \xrightarrow{g(x) = \sqrt{x-4}} [0, \infty) \xrightarrow{f(x) = \frac{1}{x^2} - 2} (-2, \infty)$

$x \neq 4$ $\qquad$ $x \neq 0$

$D = (4, \infty)$

$R = (-2, \infty)$



9. Match each graph with a formula on the left column. [A 4 marks]

[c] $f(x) = \dfrac{1}{x-4} - 2$

[ ] $f(x) = -(x-2)^2 - 4$

[a] $f(x) = -2x + 4$

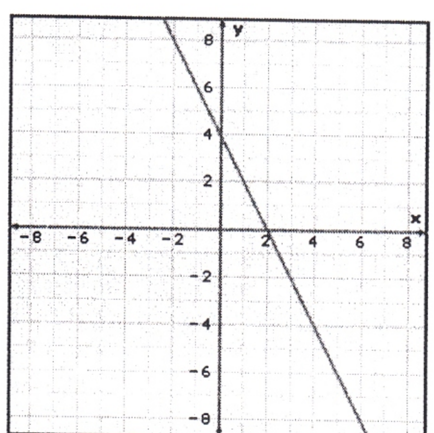
[ ] $f(x) = 4x + 2$

[d] $f(x) = -\dfrac{1}{(x-2)^2} + 3$

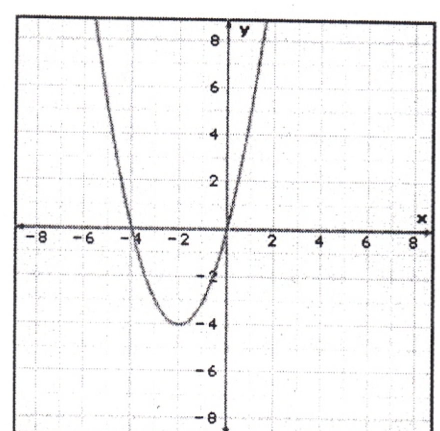
[b] $f(x) = (x+2)^2 - 4$

[ ] $f(x) = \dfrac{1}{x-2} + 3$

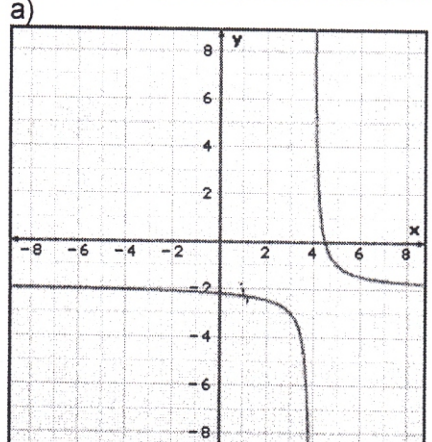
[ ] $f(x) = -\dfrac{1}{(x-4)^2} - 2$



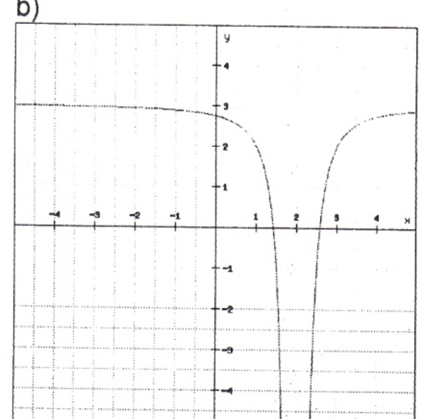
a)



b)



c)



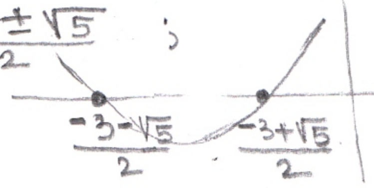
d)

10. Find the values of the parameter k such that the quadratic equation $x^2+2(k+1)x-k=0$ has two real distinct roots. [A 3 marks]

$\Delta > 0$; $[2(k+1)]^2 - 4(1)(-k) > 0$;

$4(k+1)^2 + 4k > 0$; $k^2+3k+1>0$; $k_{1/2} = \frac{-3\pm\sqrt{9-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$

$k_{1/2} = \frac{-3\pm\sqrt{5}}{2}$;



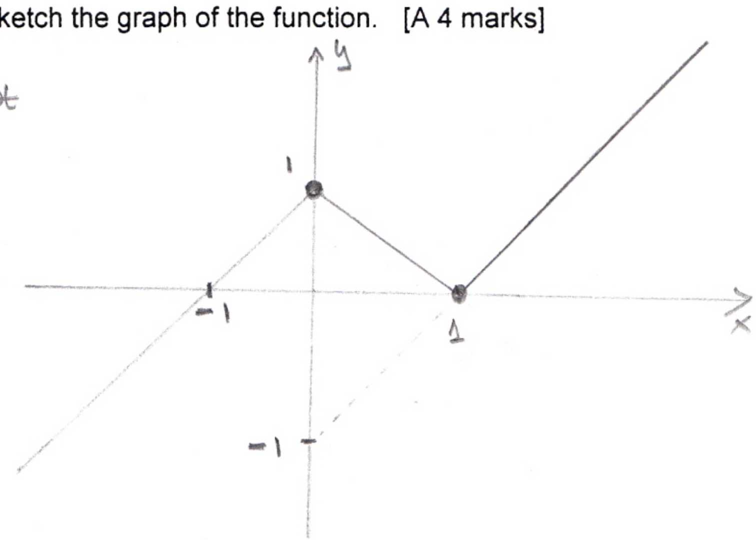
$k \in \left(-\infty, \frac{-3-\sqrt{5}}{2}\right) \cup \left(\frac{-3+\sqrt{5}}{2}, \infty\right)$

11. Express $y = x-|x|+|x-1|$ as a piecewise-defined function. Sketch the graph of the function. [A 4 marks]

$\lvert x\rvert = -x$	0	$\lvert x\rvert = x$	1	$\lvert x\rvert = x$
$\lvert x-1\rvert = 1-x$		$\lvert x-1\rvert = 1-x$		$\lvert x-1\rvert = x-1$

$$y = \begin{cases} x-(-x)+1-x, & x<0 \\ x-x+1-x, & 0\le x\le 1 \\ x-x+x-1, & x>1 \end{cases}$$

$$y = \begin{cases} x+1, & x<0 \\ 1-x, & 0\le x\le 1 \\ x-1, & x>1 \end{cases}$$



12. Consider the quadratic function $f(x)=x^2-2x-8$. Find restricted domain for this function so that the inverse relation is a function. State the domain and the range for the inverse function. Graph the original function and its inverse on the same grid. [A 4 marks]

$y=(x-1)^2-9$

$x=(y-1)^2-9$

$\pm\sqrt{x+9}=y-1$

$f^{-1}(x)=1\pm\sqrt{x+9}$

$f_1: (-\infty,1) \to (-9,\infty)$

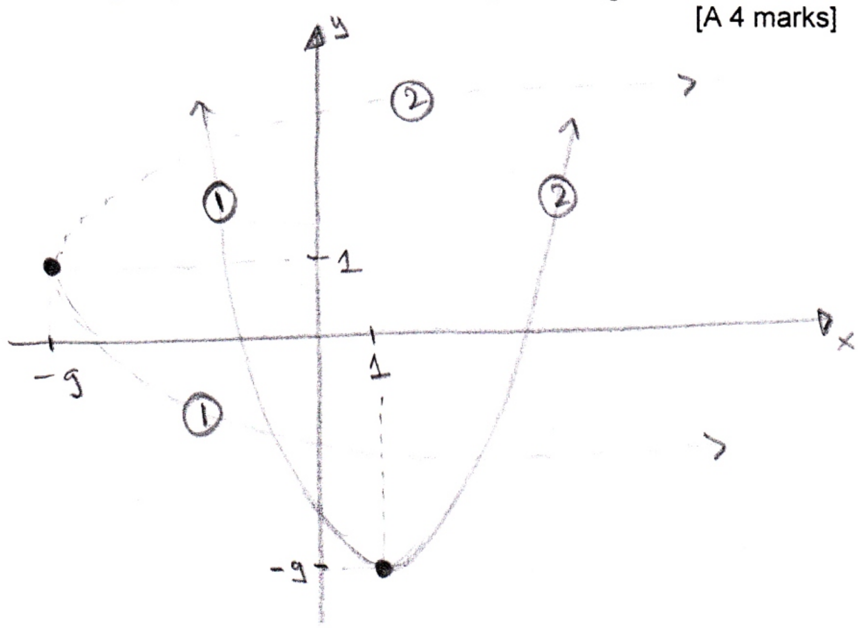
$f_1^{-1}: (-9,\infty) \to (-\infty,1)$

$f_2: [1,\infty) \to [-9,\infty)$

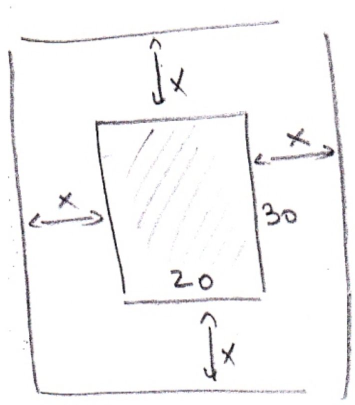
$f_2^{-1}: [-9,\infty) \to [1,\infty)$

$f_1^{-1}(x)=1-\sqrt{x+9}$

$f_2^{-1}(x)=1+\sqrt{x+9}$



13. A rectangular picture, $20cm$ by $30cm$, is to be framed with a mat such that the width of the mat is equal on all sides of the picture and the area of the mat is equal to the area of the picture. Find the width of the mat. [A 3 marks]



$20\cdot 30 = \frac{(20+2x)(30+2x)}{2}$

$1200 = 600+100x+4x^2$

$x^2+25x-150=0$

$(x-5)(x+30)=0$

$\therefore x=5$

The width of the mat is 5 cm.

14. The path of a ball is modelled by the following quadratic function $h(t) = -5t^2 + 20t + 160$ where t represents the time (in seconds) and h represents the height (in meters). [A 5 marks]

a) Determine the time when the ball hits the ground.

$t^2 - 4t - 32 = 0$

$(t-8)(t+4) = 0$

$\therefore t = 8$

the ball hits the ground after 8 s.

b) Determine the maximum height and the moment of time when the ball reaches the maximum height.

$t^* = -\frac{B}{2A} = -\frac{20}{2(-5)} = 2$

$h_{max} = h(t^*) = -5(2)^2 + 20(2) + 160 = 180\text{m}$

$\therefore$ The maximum height is 180m and is reached after 2 s.

c) Determine the time interval when the ball is at a height greater or equal to 170 m.

$h \geq 170$

$-5t^2 + 20t + 160 \geq 170$

$-5t^2 + 20t - 10 \geq 0$

$t^2 - 4t + 2 \geq 0$

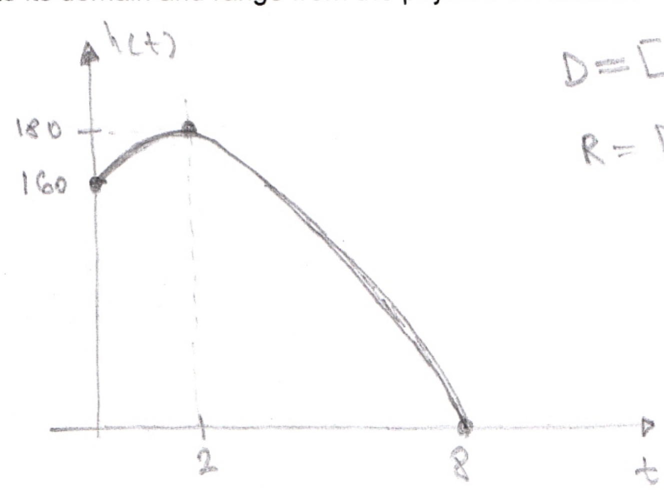
$t_{1/2} = \frac{4 \pm \sqrt{16-8}}{2}$

$= 2 \pm \sqrt{2}$

$\therefore$ The time interval is

$[2-\sqrt{2}, 2+\sqrt{2}]$

d) Graph the function $h(t)$ and state its domain and range from the physical conditions.

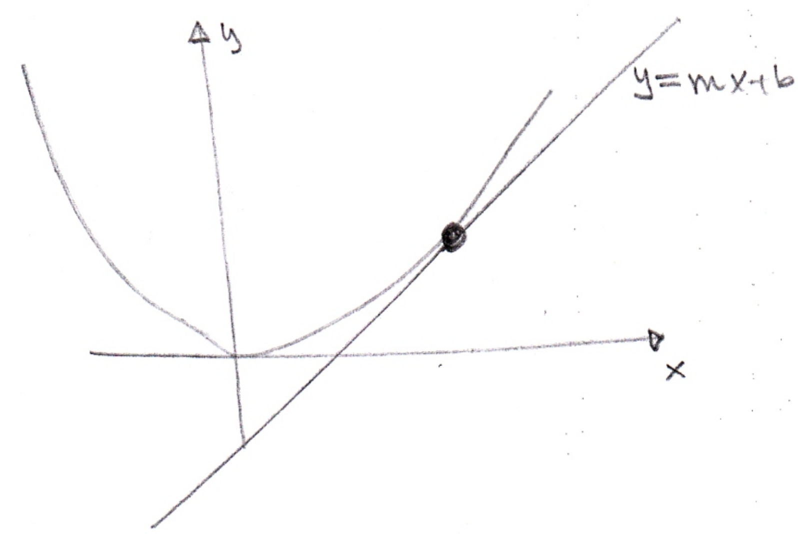


$D = [0, 8]$

$R = [0, 180]$

15. Find the equation of the line with the slope $m = 3$ that is tangent to the parabola $y = 2x^2$. [T/I 4 marks]

$\begin{cases} y = 3x + b \\ y = 2x^2 \end{cases}$



$2x^2 = 3x + b$

$2x^2 - 3x - b = 0$

$\Delta = 0$

$((-3))^2 - 4(2)(-b) = 0$

$b = -\frac{9}{8}$

$\therefore y = 3x - \frac{9}{8}$

Bonus Question. Prove that for any two one-to-one functions f and g: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[l, bend left, "f^{-1}"] \arrow[r, "g"] & z \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

$\left.\begin{array}{l} x = (f^{-1} \circ g^{-1})(z) \\ z = (g \circ f)(z) \end{array}\right\} \Rightarrow$

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$